Engineering Notes

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Course and Heading Changes in Significant Wind

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Nomenclature

a	=	acceleration
g	=	strength of gravitational acceleration
L	=	aerodynamic lift force
$p(t)$, $q(t)$, $r(t)$	=	vehicle angular rates about the x_b, y_b, z_b axes, respectively
R	=	desired orbit radius about target
R^*	=	guidance law in terms of inertial course
r	=	radius of cycloidal nonslip rotation
r^*	=	guidance law in terms of vehicle orientation
$u(t)$, $v(t)$, $w(t)$	=	components of V_a along the x_b, y_b, z_b axes
$\boldsymbol{v}$	=	measurement noise
V_a	=	airspeed, speed of vehicle relative to air mass
V_g	=	ground speed, vehicle inertial speed relative to earth surface
V_w	=	wind speed, inertial speed of the air mass
$\boldsymbol{w}$	=	process noise
Y	=	force along the body y axis
$\kappa(s)$	=	curvature of path at position s
ρ	=	radius of curvature $=\kappa^{-1}$
ϕ, θ, ψ	=	Euler orientation angles, respectively, roll or bank, pitch, and yaw or heading
χ	=	course
χ_w	=	wind direction (from convention)
ψ	=	heading, Euler yaw angle
ψ_w	=	wind vector orientation ($\psi_w = \chi_w + \pi$)
ω	=	rate of trochoid rotation $=\dot{\psi}$
${}_i\boldsymbol{\omega}_b(t)$	=	inertial angular rate vector of the vehicle
$\cdot_b$	=	referenced to the vehicle body frame
$\cdot_c$	=	commanded or desired signal
$\cdot_i$	=	referenced to the inertial frame
$\cdot_s$	=	steady state value
$\cdot_t$	=	referenced to the trochoidal frame
$\cdot_\perp$	=	oriented perpendicular to the inertial path
$\bar{\cdot}$	=	maximum value

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I. Introduction

USING straightforward differential geometry, this note presents a path formulation for maneuvering of a fixed wing aircraft in wind. The results accommodate analytical development and minimize heuristics of path-planning and decision-making algorithms. This note also makes the dynamic difference between course and heading rates explicit. This is significant for the coupling of guidance and navigation algorithms to the vehicle augmentation loops.

In this Note, wind refers to an unaccelerated horizontally moving air mass. The inertial path of a fixed wing aircraft circling in wind can be formulated as a trochoid curve and its differential geometry. This formulation has several potential benefits:

1) Analytical descriptions of velocities, direction, acceleration, arc length, and curvature;
2) Formulations for path-planning and maneuvering decision making;
3) Mapping of vehicle performance limits to the inertial path;
4) Construction of guidance algorithms that are robust with respect to wind, for example, maneuvering in winds that are stronger than the airspeed;
5) Extension of recent control theoretical analysis of multivehicle emergent behaviors.

The presence of wind differentiates heading rate of change from course rate of change. Whereas the former remains unchanged by wind, the latter is significantly affected. This is made explicit in Sec. II, the results of which are derived in Sec. III. Sections IV and V formulate these results pragmatically and indicate some consequences.

II. Yaw Rate and Course Changes

The kinematic approximation of a coordinated turn or truly banked turn [1,2] is expressed in terms of the Euler angles as

$$\dot{\psi} = \frac{g}{V_a}\tan\phi \tag{1}$$

where ψ is the aircraft heading. This expression is independent of wind, and for a constant bank angle this implies a constant heading rate of change, also called yaw rate or loosely rate of turn. In maneuvering that is referenced to the air mass, a steady wind has no effect and a standard rate of turn downwind is the same as a standard rate of turn into the wind.

In guidance and navigation, course refers to an inertially referenced direction. The course rate of change is found from kinematics as

$$\dot{\chi} = \frac{g}{V_g}\tan\phi \tag{2}$$

Expressions (1) and (2) imply that

$$\frac{\dot{\chi}}{\dot{\psi}} = \frac{V_g}{V_a} \tag{3}$$

Most guidance and navigation algorithms assume that course rate of change and yaw rate are equivalent or that the difference is addressed with inherent system robustness. Expression (3) shows that this difference can be significant, for example, if $V_w = 0.5V_a$, then the course rate of change varies from 50 to 150% of the yaw rate cyclically over each orbit.

III. Kinematics of the Coordinated Turn

A. Coordinated Turn in Terms of Airspeed

With the inertial body angular rate is expressed as ${}_i\boldsymbol{\omega}_b(t)= [p(t), q(t), r(t)]_b^T$, the lateral dynamic motion of an aircraft may be expressed in its body-fixed frame as [1]:

$$\dot{v}(t) = -r(t)u(t) - \sin\theta(t)\dot{\psi}(t)w(t) + g\sin\phi(t)\cos\theta(t) + (1/m)\{Y_{\mathrm{prop}}(t) + Y_{\mathrm{aero}}(t)\} \tag{4}$$

A coordinated turn is defined by the following conditions [2]:

1) the angular velocity of the vehicle is vertical, and

2) the sum of gravitational and centrifugal force at the center of mass is contained in the vehicle plane of symmetry.

Condition 1 gives a steady state result in kinematics, which for small vehicle pitch orientation reduces to

$$(p_s, q_s, r_s)_b^T \approx (-\theta, \sin\phi_c, \cos\phi_c)^T\dot{\psi} \tag{5}$$

Condition 2 refers to the lateral dynamics and says that the resultant force has no component along the y_b axis. For a conventional symmetric vehicle this implies that there are no resultant lateral aerodynamic or thrust forces and the lateral dynamics Eq. (4) with Eq. (5) collapses to

$$0 = -\dot{\psi}\cos\phi_c u - \theta\dot{\psi}w + g\sin\phi\cos\theta \tag{6}$$

If furthermore the vehicle operates at small aerodynamic angles, $u \gg w$ and $u \approx V_a$, then follows Eq. (1). These expressions are not affected by steady motion of the air mass.

B. Coordinated Turn in Terms of Inertial Speed

The coordinated or truly banked turn expression in terms of the inertial speed is derived from geometry as

$$ma_\perp = L\sin\phi \tag{7}$$

$$mg = L\cos\phi \tag{8}$$

The course rate of change is given by the inertial speed and curvature as

$$\dot{\chi}(t) = \kappa(t)V_g(t) \tag{9}$$

The curvature of the inertial path is related to the force perpendicular to that path as

$$a_\perp(t) = \kappa(t)V_g^2(t) \tag{10}$$

Combination of Eqs. (7–10) leads to Eq. (2).

IV. Trochoidal Path for Orbiting in Wind

The trochoid representation allows clear expressions of the above considerations. Figure 1 indicates the trochoid geometry for the inertial path of point P. Our aim is to formulate this path for arbitrary wind speed and direction, given a desired orbit radius and vehicle airspeed. In the downwind and upwind positions, respectively,

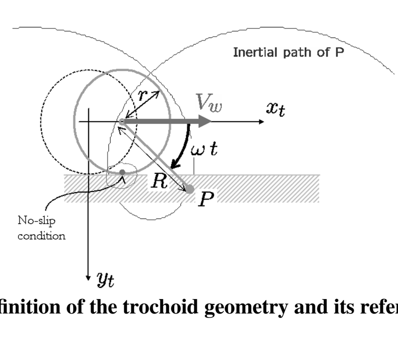

Fig. 1 Definition of the trochoid geometry and its reference frame.

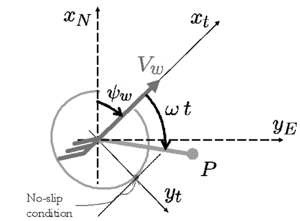

Fig. 2 Inertial orientation of the trochoid reference frame.

$$\omega(R + r) = V_a + V_w \tag{11}$$

and

$$\omega(R - r) = V_a - V_w \tag{12}$$

from which, with known wind and desired orbit radius, follows

$$\omega = \frac{V_a}{R} \quad (\text{and } \omega \equiv \dot{\psi}) \tag{13}$$

$$r = \frac{V_w}{\omega} \tag{14}$$

Thus, given the operating conditions $\{V_a, V_w, R\}$, the trochoid parameters are defined by Eqs. (13) and (14). Wind direction is used to orient the trochoid as follows.

A convenient trochoidal reference frame is defined in Fig. 2, with its inertial orientation determined by the steady wind direction. Wind direction is typically indicated as the true heading from which the air mass approaches, χ_w. It is convenient to introduce also the direction of the inertial motion of the air mass $\psi_w = \chi_w \pm \pi$. The x_t axis oriented downwind, at a heading of ψ_w from true north. The z_t-axis orientation is into the image if orbiting clockwise, or out of the image if orbiting counterclockwise. The y_t axis follows from a right-handed orientation. The position of P in the trochoidal frame can then be defined as

$$x_t(t) = R\cos(\omega t) + r\omega t + x_t(t_0) \tag{15}$$

$$y_t(t) = R\sin(\omega t) + y_t(t_0) \tag{16}$$

where

$$\omega t = \psi - \psi_w - \pi/2 \tag{17}$$

For clockwise rotation, the inertial cycloidal path is obtained as

$$\begin{pmatrix} x_N(t) \\ y_E(t) \end{pmatrix}_i = \begin{pmatrix} \cos\psi_w & -\sin\psi_w \\ \sin\psi_w & \cos\psi_w \end{pmatrix} \begin{pmatrix} x_t(t) \\ y_t(t) \end{pmatrix}_t \tag{18}$$

where the subscripts refer to north, east, inertial, and trochoidal, respectively, and ψ_w and V_w are low bandwidth estimates [3].

V. Application Examples

A. Velocity, Acceleration, Curvature, and Arc Length for Orbiting in Wind

Velocity along the trochoidal path is

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_i \begin{pmatrix} x_t \\ y_t \end{pmatrix}_t = \begin{pmatrix} r\omega - R\omega\sin\omega t \\ R\omega\cos\omega t \end{pmatrix}_t = \begin{pmatrix} V_w - V_a\sin\omega t \\ V_a\cos\omega t \end{pmatrix}_t \tag{19}$$

Inertial speed is then obtained as $V_g = \sqrt{\dot{x}_t^2(t) + \dot{y}_t^2(t)}$ or,

$$V_g^2 = R^2\omega^2 + r^2\omega^2 - 2Rr\omega^2\sin\omega t = V_a^2 + V_w^2 - 2V_aV_w\sin\omega t \tag{20}$$

The acceleration is

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_i \begin{pmatrix} x_t \\ y_t \end{pmatrix}_t = \begin{pmatrix} -R\omega^2\cos\omega t \\ -R\omega^2\sin\omega t \end{pmatrix}_t = -\frac{V_a^2}{R}\begin{pmatrix} \cos\omega t \\ \sin\omega t \end{pmatrix} \tag{21}$$

The corresponding curvature is

$$\kappa(t) = \frac{\dot{x}_t(t)\ddot{y}_t(t) - \dot{y}_t(t)\ddot{x}_t(t)}{V_g^3} \tag{22}$$

This is expressed in orbit parameters and speed ratios as

$$\kappa(t) = \frac{R^2\omega^3 - rR\omega^3\sin\omega t}{V_g^3} = \frac{(1 - V_w/V_a)}{(V_g/V_a)^3}\frac{\sin\omega t}{R} \tag{23}$$

The arc length along the inertial path is maintained by integration as

$$s(t) = \int_{t_0}^{t}\{R^2\omega^2 + r^2\omega^2 - 2Rr\omega^2\sin\omega t\}^{1/2}\,\mathrm{d}\tau \tag{24}$$

B. Left or Right Turn?

Figure 3 compares the inertial paths resulting from a 180 deg heading change into the wind and downwind. The vehicles start out heading south at $V_a = 25$ (m/s). Both vehicles then fly the same heading rate of turn in opposite directions until heading North. Wind is from 270 deg at 10 (m/s). The following list summarizes some results:

The total course change upwind ≈ 224 deg.

The total course change downwind ≈ 136 deg.

The total heading change in both cases ≈ 180 deg.

This figure is a dramatic indication that the effect of wind has important consequences for ground referenced maneuvers and for multivehicle maneuvering.

C. Consequences for Common Guidance Laws

The bandwidth of inertial guidance algorithms is affected by the magnitude of the ground speed. Guidance algorithms are often expressed as feedback of cross-track error or line-of-sight angle to a desired change in heading, which is then implemented by manipulation of vehicle bank angle. Thus, the bandwidth of the guidance logic is linked with that of the vehicle dynamics.

In the no-wind situation, a vehicle that maintains a constant airspeed is typically represented with kinematics as

$$\dot{x}_N = V_a\cos\psi \qquad \dot{y}_E = V_a\sin\psi \qquad \dot{\psi} = r^* \tag{25}$$

where r^* is a guidance law based on feedback of cross-track error or line-of-sight angle. A common example is a proportional guidance law based on the line-of-sight angle from the vehicle to a point on the desired path a constant distance l ahead of the vehicle, Fig. 4. The guidance law is then obtained as

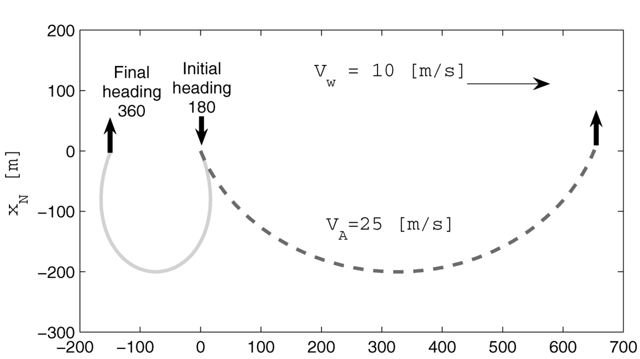

Fig. 3 Inertial paths for turns into the wind (green/solid) and down wind (red/dashed).

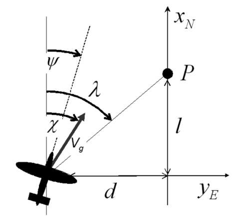

Fig. 4 Proportional guidance.

$$r^* = k_p(\lambda - \psi) \tag{26}$$

where $\lambda - \psi$ is the angle of the line of sight to P over the nose of the vehicle and

$$\lambda = \arctan\frac{-y_E}{l} \tag{27}$$

If wind is significant, guidance algorithms need to be based on feedback to a desired change in inertial course. The proportional guidance law is then obtained as

$$R^* = K_p(\lambda - \chi) \tag{28}$$

To acknowledge the effect of wind, the kinematic model in Eq. (25) should be expanded with the results of Eqs. (3) and (20). The essence of the guidance model remains the same when expressed in inertial states. This would suffice in applications where x_N, y_E, and V_g are available as measurements. However, when vehicle orientation is of concern, for example, for the payload or in analyses and design, the above must be expanded with the dynamics of vehicle orientation and inertial speed:

$$\begin{aligned} \dot{x}_N &= V_g\cos\chi \\ \dot{y}_E &= V_g\sin\chi \\ \dot{\chi} &= gV_g^{-1}\tan\phi \\ \dot{\psi} &= gV_a^{-1}\tan\phi \\ \dot{V}_g &= \dot{\chi}V_w(\cos\psi\sin\psi_w - \sin\psi\cos\psi_w) \end{aligned} \tag{29}$$

where constants V_w and ψ_w are external inputs, for example obtained from a low bandwidth observer [3]. The expression for ground speed may be obtained from Eqs. (17) and (20).

If the guidance law is based on representation Eq. (29) with guidance law Eq. (28), then the closed loop guidance dynamics are

$$\dot{\chi} = R^* \tag{30}$$

which should be implemented by inversion of Eq. (2) as

$$\phi^* = \arctan\left(\frac{V_g}{g}R^*\right) \tag{31}$$

If this is instead implemented with inversion of Eq. (1), the closed loop guidance dynamics are in effect

$$\dot{\chi} = \frac{V_a}{V_g}K_p(\lambda - \chi) \tag{32}$$

which makes explicit the well known effect of ground speed on the time constant of the guidance dynamics. In significant wind, this can lead to dynamic coupling of guidance laws and vehicle control. Furthermore, a bank angle limit (i.e., a control limit) affects the maximum heading rate as

$$\bar{\dot{\psi}} = \frac{g}{V_a}\tan\bar{\phi} \tag{33}$$

which is constant, whereas the maximum achievable course rate of change varies with ψ as

$$\dot{\bar{\chi}}(\psi) = \frac{g}{V_g(\psi)} \tan \bar{\phi} \tag{34}$$

where $V_g(\psi)$ is given by Eqs. (17) and (20), and which results in the trochoidal path of Sec. IV.

D. Formulation of a Kinematics Observer

In the implementation of many navigation algorithms, the processed output of GPS receivers is used. This output consists of inertial position, ground speed, and course, affected with some amount of measurement noise:

$$\boldsymbol{y}_g = \begin{pmatrix} x_N & y_E & V_g & \chi \end{pmatrix}^T + \boldsymbol{v} \tag{35}$$

A filter on these outputs may be formulated using kinematics in Eq. (29) as a process model

$$\dot{\boldsymbol{x}}_g = \boldsymbol{f}(\boldsymbol{x}_g, \boldsymbol{u}) + \boldsymbol{w} \tag{36}$$

where $\boldsymbol{x}_g = (x_N \quad y_E \quad \chi \quad \psi \quad V_g)^T$ and $\boldsymbol{u} = (V_a \quad V_w \quad \psi_w \quad \phi)^T$ and $\boldsymbol{w}$ is the process noise. Errors in GPS velocity estimates are typically an order of magnitude smaller than those in position. Furthermore, if the ground speed changes relatively slowly compared with the changes in position, then the above model may be reduced to

$$\begin{pmatrix} \dot{x}_N \\ \dot{y}_E \\ \dot{\chi} \end{pmatrix} = \begin{pmatrix} V_g \cos \chi \\ V_g \sin \chi \\ g V_g^{-1} \tan \phi \end{pmatrix} + \boldsymbol{w} \quad \text{and} \quad \boldsymbol{y}_g = \begin{pmatrix} x_N \\ y_E \\ \chi \end{pmatrix} + \boldsymbol{v} \tag{37}$$

with $\boldsymbol{w}$ and $\boldsymbol{v}$ redefined correspondingly. The GPS output signals can now be filtered using an output-error injection observer based on this kinematic model. With the somewhat crude assumption that $\boldsymbol{w}$ and $\boldsymbol{v}$ are zero-mean random processes, Kalman filter theory can be used to find an appropriate observer gain.

VI. Conclusions

The inertial path of a fixed wing aircraft circling in wind can be formulated as a trochoid curve and its differential geometry. This Note indicated how this formulation can accommodate analytical development and minimize heuristics of path-planning and decision-making algorithms. This will serve in the construction of guidance algorithms that are robust with respect to wind, and help to extend control-theoretical analysis of multivehicle emergent behaviors. Some examples were indicated. This Note also makes the dynamic difference between course and heading rates explicit, which serves to analyze coupling of guidance and navigation algorithms with vehicle dynamics.

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[3] Rysdyk, R., "UAV Path Following for Constant Line-of-Sight Target Observation," *Journal of Guidance, Control, and Dynamics*, Vol. 29, No. 5, 2006, pp. 1092–1100.